

GCD Computation of n Integers

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Abstract

Greatest Common Divisor (GCD) computation is one of the most important operation of algorithmic number theory. In this paper we present the algorithms for GCD computation of n integers. We extend the Euclid's algorithm and binary GCD algorithm to compute the GCD of more than two integers.

1 Introduction

Greatest Common Divisor (GCD) of two integers is the largest integer that divides both integers. GCD computation has applications in rational arithmetic for simplifying numerator and denominator of a rational number. Other applications of GCD includes integer factoring, modular arithmetic and random number generation. Euclid's algorithm is one of the most important method to compute the GCD of two integers. Lehmer [5] proposed the improvement over Euclid's algorithm for large integers. Blankinship [2] described a new version of Euclidean algorithm. Stein [7] described the binary GCD algorithm which uses only division by 2 (considered as shift operation) and subtract operation instead of expensive multiplication and division operations. Asymptotic complexity of Euclid's, binary GCD and Lehmer's algorithms remains $O(n^2)$ [4]. GCD of two integers a and b can be computed in $O((\log a)(\log b))$ bit operations [3]. Knuth and Schonhage proposed sub-quadratic algorithm for GCD computation. Stehle and Zimmermann [6] described binary recursive GCD algorithm. Sorenson proposed the generalization of the binary and left-shift binary algorithm [8]. Asymptotically fastest GCD algorithms have running time of $O(n \log^2 n \log \log n)$ bit operations [1].

In this paper we describe the algorithms for GCD computation of n integers. We extend the Euclid's algorithm to compute GCD of more than two integers as explained in [4]. We also extend the binary GCD algorithm so that it can be used to compute GCD of many integers

This paper is organized as follows. Section II describes background and motivation. Section III presents GCD algorithms for n integers and their features. Finally, section IV contains conclusion.

2 Background and Motivation

GCD of two integers a, b can be defined formally as :

$$GCD(a, b) = \max\{m \text{ such that } m|a \text{ and } m|b\}$$

GCD of two positive integers a, b such that $a > b$ can be computed using the following recursive method:

$$GCD(a, b) = GCD(b, a \bmod b)$$

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GCD operation follows the associative property, $GCD(a, GCD(b, c)) = GCD(GCD(a, b), c)$. Further we can show that

Proposition 1

$$GCD(a, b, c) = GCD(GCD(a, b), c)$$

Let d be the GCD of a, b and c . then by definition

$$\begin{aligned} d &| GCD(a, b, c) \\ &\Rightarrow d | a, b, c \\ &\Rightarrow d | a, b \text{ and } c \\ &\Rightarrow d | GCD(a, b), c \\ &\Rightarrow d | GCD(GCD(a, b), c) \end{aligned}$$

Using induction the above Proposition can be extended to a list of n integers.

Proposition 2

$$GCD(a_1, a_2, a_3, \dots, a_n) = GCD(a_1, GCD(a_2, a_3, \dots, a_n))$$

Above expression can be used to compute GCD of more than two integers. But if the list of integers (n) are very large and the pair of integers for which GCD is computed in a particular step is not randomly selected then the computation can be expensive.

In this paper we discuss the algorithms for GCD computation of many integers by actually extending the properties of the GCD computation of two integers.

3 GCD Algorithms

3.1 Extension to Euclid's Algorithm

Euclid's algorithm for GCD of two integers can be extended to GCD of n integers (in this paper integers are non-negative unless otherwise specified) using the following facts:

Let a_1, a_2, \dots, a_n be n integers.

Proposition 3

$$GCD(0, 0, \dots, 0) = 0$$

This can be established by taking the convention $GCD(0, 0)$ for $n = 2$ and extending it to n numbers.

Proposition 4

$$GCD(a_1, 0, \dots, 0) = a_1$$

Proof: It follows from the fact $GCD(a_1, 0) = a_1$ together with associativity of GCD.

$$GCD(a_1, 0, \dots, 0)$$

$$= GCD(a_1, GCD(0, 0, \dots, 0)) = GCD(a_1, 0) = a_1$$

or, $GCD(\dots GCD(GCD(a_1, 0), 0) \dots, 0) = a_1$

Proposition 5

Let $\min(a_1, a_2, \dots, a_n) = a_1$ then

$$GCD(a_1, a_2, \dots, a_n) = GCD(a_1, a_2 \bmod a_1, \dots, a_n \bmod a_1)$$

Proof: Note that if $a_1 < a_2$ then $GCD(a_1, a_2) = GCD(a_1, a_2 - xa_1)$ for any integer x . As any common divisor a_1 and a_2 will be divisor of both a_1 and $a_2 - xa_1$, and also any common divisor of a_1 and $a_2 - xa_1$ will divide both a_1 and a_2 . Now, Proposition 5 follows for $n = 2$. Similarly, it can be extended using induction for n numbers.

GCD-N is described in Algorithm 1. In this algorithm if at any time there is only one non-zero a_i , it will be considered as GCD, in other cases all a_i 's will be reduced by $a_i \bmod a_j$, where a_j is the least non-zero integer at a given iteration. Detailed description of GCD-N is as follows: first while loop is used to check that at least one a_i is non zero. If all numbers are 0, final GCD will be taken as 0. First for loop is used to store least non-zero integer in a_1 . Second for loop is used to reduce other a_i 's to $a_i \bmod a_1$ except a_1 . Third for loop is used to store largest and second largest integers in a_n and a_{n-1} respectively. Finally, if loop is used to check whether a_n is the only non-zero integer, if this is the case, a_n is returned otherwise while loop is called again.

Correctness of GCD-N algorithm follows from the Proposition 3, 4 and 5.

Example 1: Let $a_1 = 22, a_2 = 36, a_3 = 74, a_4 = 98$ then

$$\begin{aligned} &GCD(a_1, a_2, a_3, a_4) \\ &\Rightarrow GCD(22, 36, 74, 98) \\ &\Rightarrow GCD(22, 36 \bmod 22, 74 \bmod 22, 98 \bmod 22) \\ &\Rightarrow GCD(22, 14, 8, 10) \\ &\Rightarrow GCD(8, 14, 22, 10) \\ &\Rightarrow GCD(8, 14 \bmod 8, 22 \bmod 8, 10 \bmod 8) \\ &\Rightarrow GCD(8, 6, 6, 2) \\ &\Rightarrow GCD(2, 6, 6, 8) \\ &\Rightarrow GCD(2, 6 \bmod 2, 6 \bmod 2, 8 \bmod 2) \\ &\Rightarrow GCD(2, 0, 0, 0) \\ &\Rightarrow 2 \end{aligned}$$

3.2 Extension to Binary GCD Algorithm

GCD of n integers can be computed using only shift and subtract operation similar to binary GCD algorithm. This algorithm is based on the following facts.

Proposition 6

Let a_1, a_2, \dots, a_n be even integers then

$$GCD(a_1, a_2, \dots, a_n) = 2 \cdot GCD(a_1/2, a_2/2, \dots, a_n/2)$$

Proof: Consider the above statement for $n = 2$. Let $a'_1 = a_1/2$ and $a'_2 = a_2/2$. By definition $GCD(a_1, a_2)$ is least positive value of $a_1m_1 + a_2m_2$ where m_1 and m_2 range over all integers. Now, $GCD(a_1, a_2) = GCD(2a'_1, 2a'_2)$
 \Rightarrow least positive value of $2a'_1m_1 + 2a'_2m_2$

Algorithm 1 : GCD-N (a_1, a_2, \dots, a_n)

INPUT: a_1, a_2, \dots, a_n
OUTPUT: $GCD(a_1, a_2, \dots, a_n)$
while (a_1 or a_2 or ... or a_n) **do**
 for ($i \leftarrow 2; i \leq n; i \leftarrow i + 1$) **do**
 if ($a_1 > a_i$ and $a_i \neq 0$) **then**
 $temp \leftarrow a_i$
 $a_i \leftarrow a_1$
 $a_1 \leftarrow temp$
 end if
 end for
 for ($i \leftarrow 2; i \leq n; i \leftarrow i + 1$) **do**
 $a_i \leftarrow a_i \bmod a_1$
 end for
 for ($i \leftarrow n - 2; i \geq 1; i \leftarrow i - 1$) **do**
 if ($a_i > a_n$) **then**
 $temp \leftarrow a_i$
 $a_i \leftarrow a_n$
 $a_n \leftarrow temp$
 end if
 if ($a_i < a_n$ and $a_i > a_{n-1}$) **then**
 $temp \leftarrow a_i$
 $a_i \leftarrow a_{n-1}$
 $a_{n-1} \leftarrow temp$
 end if
 end for
 if ($a_n \neq 0$ and $a_{n-1} == 0$) **then**
 return a_n
 end if
end while

$$\Rightarrow 2.\{\text{least positive value of } a'_1 m_1 + a'_2 m_2\}$$

$$\Rightarrow 2.GCD(a'_1, a'_2)$$

$$\Rightarrow 2.GCD(a_1/2, a_2/2)$$

Therefore, above statement is proved for $n = 2$. Now, using induction the statement can be shown to hold for n .

Alternatively one can use the general definition:

$$GCD(a_1, a_2, \dots, a_n) = \prod_{p \text{ prime}} p^{\min(a_{1p}, a_{2p}, \dots, a_{np})} \quad (1)$$

Where each a_i is expressed in its unique prime factorization in ascending order.

$$a_i = 2^{a_{i2}}.3^{a_{i3}}.5^{a_{i5}} \dots = \prod_{p \text{ prime}} p^{a_{ip}}$$

Now, using the Equation 1, Proposition 6 can be easily established. Since a_1, a_2, \dots, a_n are all even integers. Let $a'_1 = a_1/2, a'_2 = a_2/2, \dots, a'_n = a_n/2$ then

$$\begin{aligned} & GCD(a_1, a_2, \dots, a_n) \\ & \Rightarrow GCD(2a'_1, 2a'_2, \dots, 2a'_n) \\ & \Rightarrow 2.GCD(a'_1, a'_2, \dots, a'_n) \\ & \Rightarrow 2.GCD(a_1/2, a_2/2, \dots, a_n/2) \end{aligned}$$

Proposition 7

Let a_1, a_2, \dots, a_m be odd and $a_{m+1}, a_{m+2}, \dots, a_n$ be even integers then

$$\begin{aligned} & GCD(a_1, a_2, \dots, a_m, \dots, a_n) = \\ & GCD(a_1, \dots, a_m, a_{m+1}/2, \dots, a_n/2) \end{aligned}$$

Proof: Since a_1, a_2, \dots, a_m are odd integers, they are not divisible by 2. But $a_{m+1}, a_{m+2}, \dots, a_n$ are even, and hence divisible by 2. Let $a'_{m+1} = a_{m+1}/2, a'_{m+2} = a_{m+2}/2, \dots, a'_n = a_n/2$. Then using the Equation 1, again:

$$\begin{aligned} & GCD(a_1, a_2, \dots, a_m, a_{m+1}, a_{m+2}, \dots, a_n) \\ & \Rightarrow GCD(a_1, a_2, \dots, a_m, 2a'_{m+1}, 2a'_{m+2}, \dots, 2a'_n) \\ & \Rightarrow GCD(a_1, a_2, \dots, a_m, a'_{m+1}, a'_{m+2}, \dots, a'_n), \text{ Since 2 is not common factor} \\ & \Rightarrow GCD(a_1, a_2, \dots, a_m, a_{m+1}/2, a_{m+2}/2, \dots, a_n/2) \end{aligned}$$

Proposition 8

Let a_1, a_2, \dots, a_n be odd integers

Let $\min(a_1, a_2, \dots, a_n) = a_1$ then

$$GCD(a_1, a_2, \dots, a_n) = GCD(a_1, (a_2 - a_1)/2, \dots, (a_n - a_1)/2)$$

Proof: Using the explanation in Proposition 5, if $a_1 < a_2$ then $GCD(a_1, a_2) = GCD(a_1, a_2 - xa_1)$, by putting $x = 1$ and extending it to the case of n integers, we can write.

$$GCD(a_1, a_2, \dots, a_n) = GCD(a_1, (a_2 - a_1), \dots, (a_n - a_1))$$

Now all terms $(a_2 - a_1), (a_3 - a_1), \dots, (a_n - a_1)$ are even except first term a_1 which is still odd.

Therefore we can use the Proposition 7 to further reduce it. $GCD(a_1, (a_2 - a_1), \dots, (a_n - a_1))$

$$= GCD(a_1, (a_2 - a_1)/2, \dots, (a_n - a_1)/2)$$

BINARY-GCD-N is described in Algorithm 2. In this algorithm first while loop is used to check how many iterations all of the n integers are divisible by 2, and this value is stored in counter p . Second while loop is used to check for all integers to be not zero. First for loop is used to store least non-zero integer in a_1 . Second for loop is used to reduce other a_i 's to $a_i - a_1$ except a_1 , and this process is repeated until all a_i 's except one are zero. Correctness of BINARY-GCD-N algorithm follows from Proposition 6,7 and 8.

Algorithm 2 : BINARY-GCD-N (a_1, a_2, \dots, a_n)

INPUT: a_1, a_2, \dots, a_n
OUTPUT: $GCD(a_1, a_2, \dots, a_n)$
while ($a_1 \bmod 2 == 0$ and $a_2 \bmod 2 == 0$ and ... and $a_n \bmod 2 == 0$) **do**
 $a_1 \leftarrow a_1/2$
 $a_2 \leftarrow a_2/2$

 $a_n \leftarrow a_n/2$
 $p \leftarrow p + 1$
end while
while (a_2 or a_3 or ... or a_n) **do**
 while ($a_1 \bmod 2 == 0$) **do**
 $a_1 \leftarrow a_1/2$
 end while
 while ($a_2 \bmod 2 == 0$) **do**
 $a_2 \leftarrow a_2/2$
 end while

 while ($a_n \bmod 2 == 0$) **do**
 $a_n \leftarrow a_n/2$
 end while
 for ($i \leftarrow 2; i \leq n; i \leftarrow i + 1$) **do**
 if ($a_1 > a_i$) **then**
 $temp \leftarrow a_1$
 $a_1 \leftarrow a_i$
 $a_i \leftarrow temp$
 end if
 end for
 for ($i \leftarrow 2; i \leq n; i \leftarrow i + 1$) **do**
 $a_i \leftarrow a_i - a_1$
 end for
end while
return $a_1 * 2^p$

Example 2: Let $a_1 = 14, a_2 = 28, a_3 = 56, a_4 = 98$ then
 $GCD(a_1, a_2, a_3, a_4)$
 $\Rightarrow GCD(14, 28, 56, 98)$
 $\Rightarrow 2.GCD(7, 14, 28, 49)$
 $\Rightarrow 2.GCD(7, 7, 14, 49)$
 $\Rightarrow 2.GCD(7, 7, 7, 49)$

$\Rightarrow 2.GCD(7, 0, 0, 42)$
 $\Rightarrow 2.GCD(7, 0, 0, 21)$
 $\Rightarrow 2.GCD(7, 0, 0, 14)$
 $\Rightarrow 2.GCD(7, 0, 0, 7)$
 $\Rightarrow 2.GCD(7, 0, 0, 0)$
 $\Rightarrow 2.7 = 14$

4 Conclusion and Future Work

In this paper we have presented the algorithms for GCD computation of many integers. Traditional method of computing GCD of many integers by recursively calling Euclid's for each pair of integers can be expensive if the list of integers are not selected randomly. Future work can be to extend these algorithms to GCD computation of n polynomials.

References

- [1] Aho, A.V., Hopcroft, J.E., Ullman, J.D., "The design and analysis of computer algorithms", Addison-Wesley, 1974.
- [2] Blankinship, W.A., "A new version of the Euclidean algorithm", Amer. Math. Mon. 70, 1963.
- [3] Brent, R.P., Zimmerman, P., "Modern computer arithmetic", Cambridge University Press, 2010.
- [4] Knuth, D.E., "The art of computer programming, vol. 2, seminumerical algorithms", Addison-Wesley, 1998.
- [5] Lehmer, D.H., "Euclid's algorithm for large numbers", Amer. Math. Mon. 45, 1938.
- [6] Stehle, D., Zimmermann, P., "A binary recursive GCD algorithm", In Proc. of the International Symposium on Algorithmic Number Theory, 2004.
- [7] Stein, J., "Computational problems associated with Racah algebra", Journal of Computational Physics 1 no.3 , 1967.
- [8] Sorenson, J. P., "Two fast GCD algorithms", Journal of Algorithms 16, 1994.